IDENTIFICATION OF THE NONLINEAR VISCOUS PROPERTIES OF FLUIDS BY THE VIBRATIONAL METHOD

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Effects occurring in an oscillatory viscosimeter upon its filling with nonlinear viscous fluids whose rheological properties are described by the Ostwald–Weyl model have been revealed. The possibilities of the vibrational method for identification of the rheological belonging of media to nonlinear viscous ones with a power rheological law and of their nonlinear properties in the regime of free damped and forced vibrations have been discussed.

Introduction. Non-Newtonian fluids are the basic working media in diverse technological processes. Most experiments on studying the rheological properties of high-temperature and chemically aggressive liquids, such as, for example, metal and slag melts, have been interpreted under the assumption of a Newtonian behavior of their flow; this can lead to contradictions in the value of the viscosity and in the character of its dependence on thermodynamic parameters. The vibrational method [1] possesses a number of advantages over the remaining methods in studying the characteristics of such media which are difficult to investigate. We recall that here the properties of a fluid are judged from the parameters of forced vibrations of a probe of arbitrary geometry, for example, a plate, immersed in this medium. The problem is to predict the law of vibrations of the plate and it is conjugate: the motion of the probe is closely related to the medium's motion excited by it.

Such a method is characterized by the time variation in the increments of stresses and strains, which makes it possible to find, for example, the elastic properties of liquid media or the properties of fluid systems with a variable ratio between the stress and the rate of shear. Moreover, one can realize both the minimum total strains and low strain rates in the regime of damped vibrations and can find, in particular, weakly plastic properties. Thus, the conditions realized in an oscillatory viscosimeter enable one to make individual non-Newtonian effects observable for fluids that are usually accepted as Newtonian.

Flows excited in a non-Newtonian medium by a plate oscillating in its plane have long attracted the attention of researchers. However, all these works have been devoted to solution of a nonconjugate problem where the law of motion of the plate is prescribed, for example, by a harmonic time function, and they refer mainly to viscoelastic fluids: Oldroyd–B media [2], those of Johnson and Segalman [3], Rivlin–Ericksen media of second and third orders, etc. (see, for example, [4–6]). Procedures for evaluating the properties of non-Newtonian media by the vibrational method are absent. At present, we know of results concerning only the simplest types of viscoelastic media, most frequently linear ones, in addition to a Newtonian medium, i.e., when it is possible to quite easily obtain the analytical expression for the law of vibrations of a plate in a regular steady-state regime. For non-Newtonian media this law is generally different from the harmonic law. Also, it is noteworthy that the vibrational method has been developed for the regime of forced vibrations. The possibility of using it in the regime of free damped oscillations has been noted, among other works, in [1] in connection with the measurement of the properties of Newtonian media with low viscosities, but a correct substantiation of calculated relations is absent.

The present work seeks to investigate the laws of motion of a plate executing forced or free damped vibrations in a nonlinear viscous medium and to evaluate the possibilities of the vibrational method for identification of its rheological type and properties by the parameters (observed in experiment) of these vibrations using a fluid with the Ostwald–Weyl rheological law as an example.

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Mathematical Formulation of the Problem and Method of Solution. A mathematical model of viscosimetric experiments in the case of forced vibrations will be represented in the following form:

(1) the equation of motion of the plate

$$\frac{d^2\xi}{dT^2} + \xi = \sin\frac{T}{\lambda} - \Phi_{\text{fr}}\,; \tag{1}$$

(2) the equation of motion of the fluid

$$\frac{\partial U}{\partial T} = \frac{\partial \sigma_{\zeta\xi}}{\partial \zeta}\,; \tag{2}$$

(3) the initial boundary conditions for (1) and (2)

$$\left.\frac{d\xi}{dT}\right|_{T=0} = 0\,, \quad \xi\,(0) = 0\,, \quad U\,(\zeta, 0) = 0\,, \quad U\,(0, T) = y\,\frac{d\xi}{dT}\,, \quad U\,(\infty, T) = 0\,; \tag{3}$$

(4) the rheological equation of state according to the Ostwald–Weyl model

$$\sigma_{\zeta\xi} = bD_{\zeta\xi}D^{m-1}\,, \tag{4}$$

where

$$b = \omega_0^{m-1}\frac{K}{\nu\rho}\,; \quad \xi = \frac{xk}{F}\,; \quad T = \omega_0 t\,; \quad \lambda = \frac{\omega_0}{\omega}\,; \quad \omega_0^2 = \frac{k}{M}\,; \quad \zeta = \frac{z}{d}\,; \quad d = \sqrt{\frac{\nu}{\omega_0}}\,;$$

$$\Phi_{\text{fr}} = -\,2A\sigma_{\zeta\xi}\Big|_{\zeta=0}\,; \quad A = S\nu\rho\,\frac{\omega_0}{F}\,; \quad y = \frac{F}{kd}\,; \quad \beta = \sqrt{2}\,Ay\,; \quad U = \frac{V}{d\omega_0}\,; \quad D_{\zeta\xi} = \frac{\partial U}{\partial \zeta}\,. \tag{5}$$

Here we disregard the damping of vibrations in the absence of a medium and the edge effects. For the flow conditions considered we have $D = \left|D_{\zeta\xi}\right|$. To solve the system of nonlinear equations (1)–(4) we have used the method of lines. The resulting system of ordinary differential equations was integrated, in particular, by the Runge–Kutta method of fourth order with accuracy control and automatic selection of the time step, by the Adams method of fifth order of accuracy in the Nordsieck form, and others. To integrate rigid systems occurring under certain experimental conditions we used the Gear method of sixth order of accuracy. The derivatives with respect to coordinate were approximated by difference relations with five nodal points, ensuring the accuracy of the fourth order of the step over the coordinate.

For the regime of damped vibrations the experimental model will have a form analogous to (1)–(5), with the only exception being that the equation of motion of the plate will be written as

$$\frac{d^2\xi}{dT^2} + \xi = -\,\Phi_{\text{fr}}\,; \tag{6}$$

the initial boundary conditions will be

$$\left.\frac{d\xi}{dT}\right|_{T=0} = 0\,, \quad \xi\,(0) = \xi_0\,, \quad U\,(\zeta, 0) = 0\,, \quad U\,(0, T) = \frac{d\xi}{dT}\,, \quad U\,(\infty, T) = 0\,, \tag{7}$$

where the parameters (5) will now be

$$b = \omega_0^{m-1}\frac{K}{\nu\rho}\,; \quad \xi = \frac{x}{d}\,; \quad T = \omega_0 t\,; \quad \lambda = \frac{\omega_0}{\omega}\,; \quad \omega_0^2 = \frac{k}{M}\,; \quad \zeta = \frac{z}{d}\,; \quad d = \sqrt{\frac{\nu}{\omega_0}}\,;$$

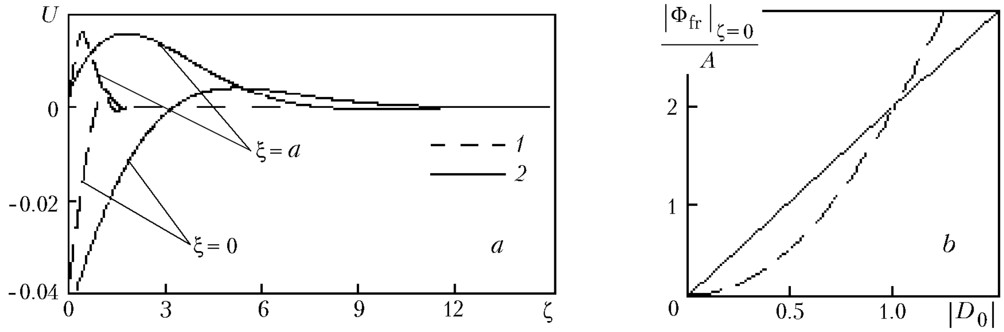

Fig. 1. On the dependence $U = U(\zeta)$ for the dilatant medium with low rates of shear ($b = 1$, $\beta = 1$, $y = 0.1$, and $\lambda = 3$): 1) $m = 2$; 2) 1.

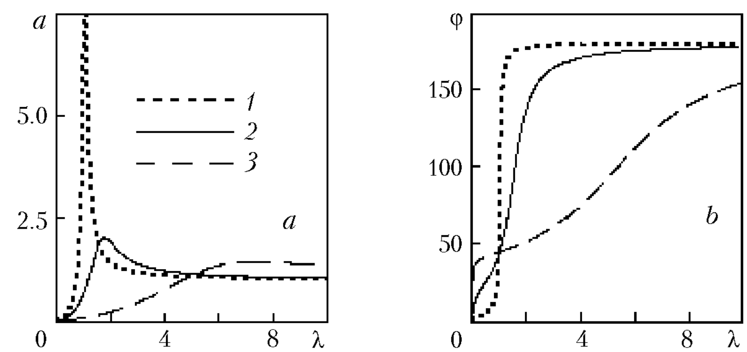

Fig. 2. Amplitude (a) and phase (b) characteristics for the Newtonian medium: 1) $\beta = 0.1$; 2) 1.0; 3) 10.

$$\Phi_{\rm fr} = -2A\sigma_{\zeta\xi}\Big|_{\zeta=0}\,;\quad A = Sd\,\frac{\rho}{M}\,;\quad \beta = \sqrt{2}A\,;\quad U = \frac{V}{d\omega_0}\,;\quad D_{\zeta\xi} = \frac{\partial U}{\partial \zeta}\,. \qquad (8)$$

Results and Discussion. I. Method of Forced Vibrations. *Distinctive Features of Fluid Flow on the Plate.* For nonlinear viscous media we can recognize two types of flow: with a viscosity higher and lower than the Newtonian one, depending on which the boundary of the region of developed flow approaches the plate or moves from it as compared to the Newtonian fluid (Fig. 1). The reason is that the penetration depth is in proportion to the apparent viscosity bD^{m-1}, determined by the ratio of the stress and the rate of shear, which decreases with growth in m and decrease in D under given experimental conditions (when $D < 1$) for dilatant media. Thus, Fig. 1a corresponds to small D ($|D_0| < 0.1$). The flow curve for $m = 2$ in Fig. 1b runs below the straight line for the Newtonian medium with $m = 1$, and the boundary of the region, where $U \sim 0$ in Fig. 1b, is closer to the plate for the medium with $m = 2$. For β = const, the range of the rates of shear passed in the process of vibrations is extended with growth in y, i.e., with increase, for example, in the amplitude of the driving force F, and for $|D_0| >> 0.1$ the boundary is substantially farther from the plate than for the Newtonian fluid. What this can mean in practice is that the assumption of the infinity of the medium is no longer acceptable and the influence of the walls cannot be disregarded.

Amplitude-Phase Characteristics. The amplitude and phase characteristics of motion have been studied for Ostwald–Weyl fluids (Figs. 2–4). The results obtained for $m = 1$ are consistent with those given in [1] if, unlike [1], we use the expression for the additional mass in the form $\mu = S\rho\sqrt{2\nu/\omega}$. In this case, the amplitude and phase characteristics for different values of β have the form presented in Fig. 2a and b. Figure 3 shows the dependence of the phase shift of the system's vibrations from the phase of the exciting force on the frequency, whereas Fig. 4 shows the dependence of the amplitude of steady-state vibrations on the experimental conditions and the properties of nonlinear viscous fluids. For Newtonian fluids the amplitude curves depend only on the dimensionless viscosity β (Fig. 2a), but for nonlinear viscous media for β = const and y = var they are exposed to shear (Fig. 4d), which enables us to identity their non-Newtonian behavior.

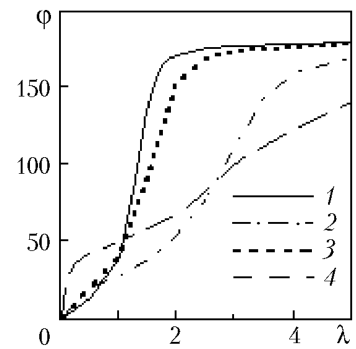

Fig. 3. Dependence $\varphi = \varphi(\lambda)$ for Ostwald–Weyl media (y = 0.1): 1) m = 2, b = 1, and β = 1; 2) 2, 1, and 10; 3) 2, 10, and 1; 4) 0.5, 1, and 1.

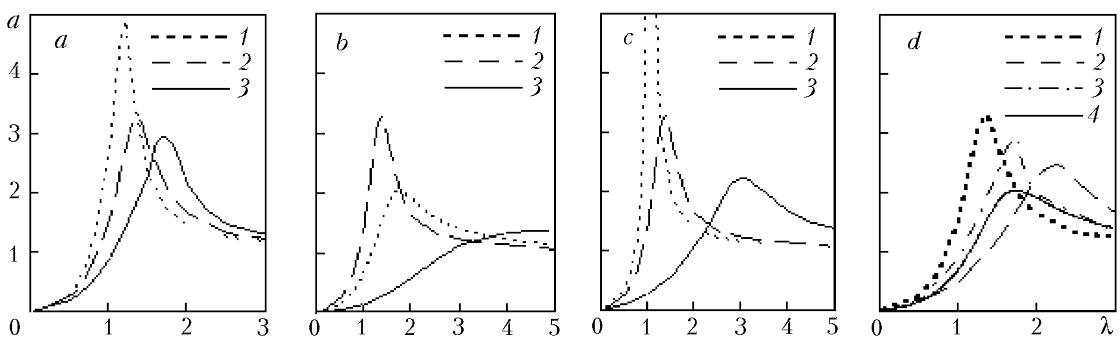

Fig. 4. Dependence of the form of resonant curves on the experimental conditions and the medium's properties: a) 1) b = 0.1, 2) 1, 3) 10 (m = 2, β = 1, and y = 0.1); b) 1) m = 1, 2) 2, 3) 0.5 (b = 1, β = 1, and y = 0.1); c) 1) β = 0.1, 2) 1; 3) 10 (b = 1, m = 2, and y = 0.1); d) y = 0.1, 2) 10, 3) 1; 4) m = 1 (b = 1, m = 2, and β = 1).

Identification of the Unknown Rheological Properties. On the basis of a qualitative pattern (Figs. 3 and 4), we can propose a few methods of evaluation of the rheological parameters of nonlinear viscous media. We demonstrate one method using a medium with m = 2 as an example. We set b = 1, i.e., take the viscosity $\nu = \omega_0^{m-1}K/\rho$ in the parameters d, A, and others (5). The product $y\beta$ is independent of ν, and for $y\beta$ = const, y = var, and a certain m we obtain a series of resonant curves corresponding to different ν with certain resonant frequencies $\lambda_{\rm res}$ and amplitudes $a_{\rm res}$ (see, for example, Fig. 2a for m = 1, where, in particular, we have the values y = 10, 1.0, and 0.1 for $y\beta$ = 1). In this case, the exponent m is conveniently determined from the dependence $a_{\rm res} = a_{\rm res}(\lambda_{\rm res})$ and, in particular, from the quantity $a_{\rm as}$ for higher values of $\lambda_{\rm res}$, when $da_{\rm res}/d\lambda_{\rm res} \sim 0$ (Fig. 5a). It is noteworthy that the function $a_{\rm as} = a_{\rm as}(m)$ is monotonically increasing.

Evaluation of m and K can be performed on the basis of the behavior of the curves $a_{\rm res} = a_{\rm res}(\beta)$ and $\lambda_{\rm res} = \lambda_{\rm res}(\beta)$ (Fig. 5b and c). Thus, the curve corresponding to m = 1 in Fig. 5b asymptotically tends to a straight line with an angular coefficient $1/\beta$. When m is known, we can evaluate the constant K, setting ν in (5) equal to a certain constant and determining b for prescribed β and y = const from the amplitude characteristics analogous to those in Fig. 4a. Here the discussion of curves of the form $\lambda_{\rm res} = \lambda_{\rm res}(b)$ and $a_{\rm res} = a_{\rm res}(b)$ is of interest.

Spectra of Kinematic and Dynamic Characteristics. For the regime of steady-state vibrations we have constructed frequency spectra for the rate of shear D_0 (Fig. 6) and the friction force on the plate (these spectra are qualitatively analogous to those given in Fig. 6). The appearance of the odd harmonics of the driving force has been noted; the harmonics were found earlier in [7] in a simplified model with lumped parameters. The intensity distribution of spectral peaks, including the form of their envelope curve, is determined by the nonlinear properties of the fluid and

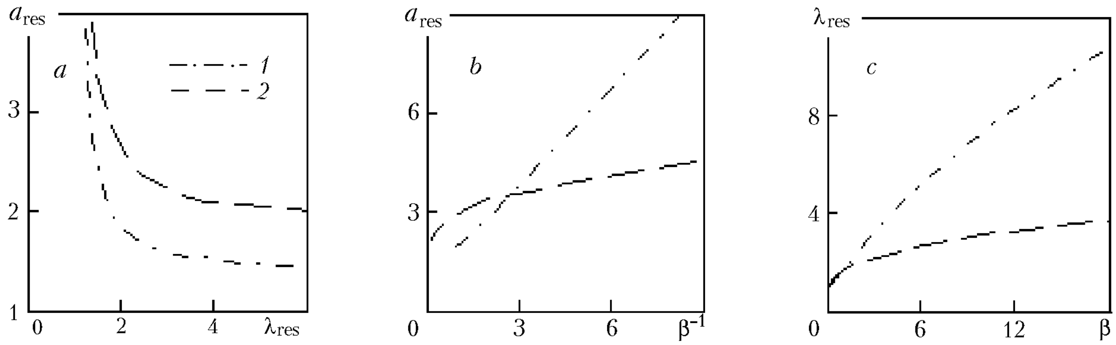

Fig. 5. On the theory of the method of evaluation of the constant and the exponent of the power law ($y\beta = 1$): 1) $m = 1$; 2) 2.

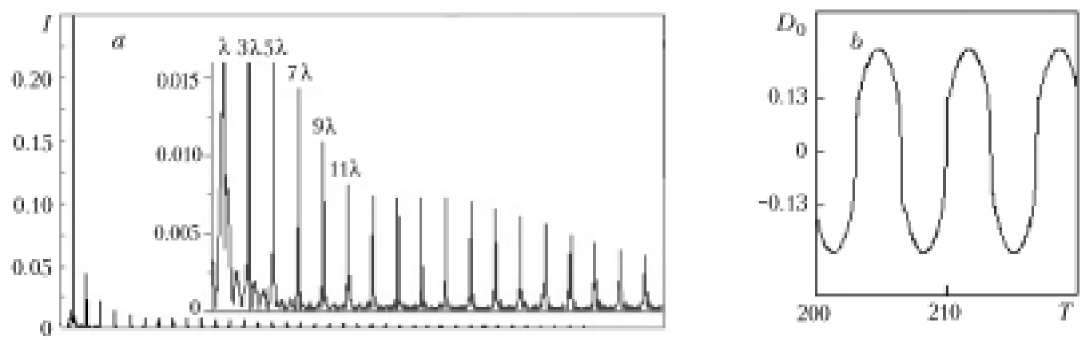

Fig. 6. Spectrum (a) and the dependence of the rate of shear of the plate on the time (b) ($b = 1$, $\beta = 1$, $m = 3$, $\lambda = 1.1$, and $y = 0.01$).

by the experimental conditions, which enables us to investigate the properties of such media within the framework of Fourier rheology.

In the parametric range indicated in Figs. 3–6, the nonlinearity exerts no substantial influence on the harmonicity of the law of vibrations. A weak manifestation of odd harmonics (3λ, 5λ, ...) in realization of transient processes is possible in the dependence $\xi = \xi(T)$. With increase in the contribution of the friction force to the resulting force, growth in the vibration amplitude, etc., we have conditions contributing to the manifestation of such nonlinear effects. Thus, for example, for larger y and other conditions of Fig. 6, additional odd harmonics, apart from the harmonics of the fundamental frequency of the driving force, begin to appear in the dependence $\xi = \xi(T)$ for the values $\lambda_{res} \sim \lambda_{res}$; in particular, in realization of a stable computational algorithm, there can occur satellites with odd harmonics of the driving force near these peaks on the vibration spectrum.

II. Method of Damped Vibrations. *Viscosimetric Equation for a Newtonian Medium.* We primarily consider the particular case of a Newtonian medium where $b = m = 1$. Seeking the law of vibrations of a plate in the form

$$\xi = \xi_0 \exp\left[-iT(\theta - \Delta i)\right] , \tag{9}$$

by solution of system (2), (4), and (6)–(8), we find the dependence for determination of the parameters of vibrations

$$[1 - (\theta - \Delta i)^2] - 2A\sqrt{i}\,(\theta - \Delta i)^{3/2} = 0 , \tag{10}$$

where $\theta = \omega/\omega_0 = 1/\lambda$ and $\Delta = (\delta/2\pi)\omega/\omega_0$, $\omega = 2\pi/\tau$, and $\omega_0 = 2\pi/\tau_0$.

The vibration parameters θ and Δ for the Newtonian medium are independent of the initial vibration amplitude ξ_0 and are determined by a single parameter A. High values of δ bound the range of expedient A values, for example, to $A < 0.1$. Steady-state vibrations of the plate immersed in such a fluid are isosynchronous. Such an asymptotic regime can be disturbed for non-Newtonian media. In what follows, by the vibration parameters we will mean the quantities

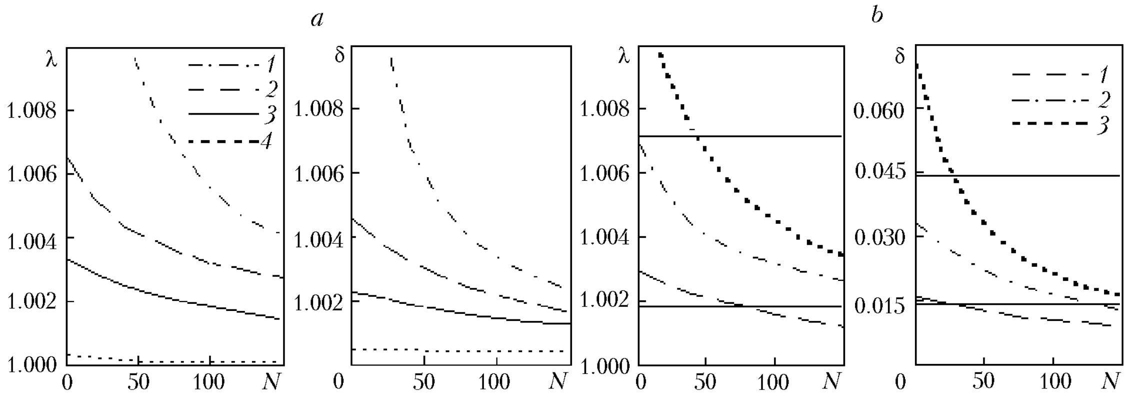

Fig. 7. Variation in the parameters of vibrations in the process of their damping for different values of the parameters A (a) and b (b): a) 1) A = 0.05, 2) 0.01, 3) 0.005, 4) 0.001 (b = 1 and m = 2); b) 1) b = 0.1, 2) 1, 3) 10 (A = 0.01 and m = 2).

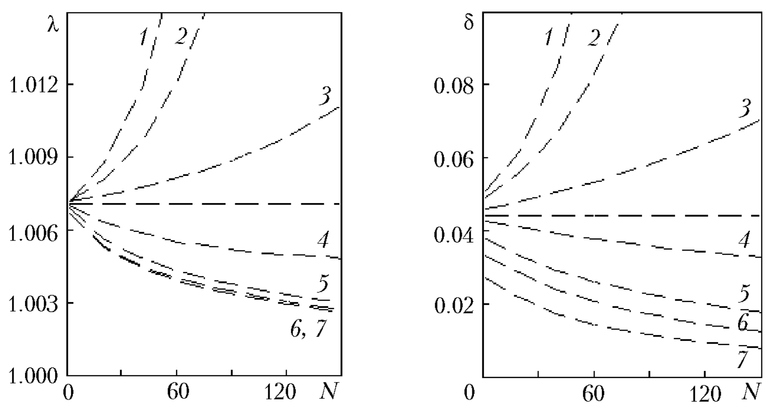

Fig. 8. Variation in the parameters of vibrations in the process of their damping for different m: 1) m = 2/3; 2) 3/4; 3) 0.9; 4) 1.1; 5) 3/2; 6) 2; 7) 3 (b = 1 and A = 0.01).

$$\tau = 2\Delta t\,, \quad \delta = 2\ \ln \left|\xi_1/\xi_2\right|\,. \tag{11}$$

Distinctive Features of Viscosimeter Motion. The dependence of the behavior of the vibration parameters on time, i.e., on the vibration No. N, have been demonstrated in Figs. 7 and 8 for Ostwald–Weyl media under different experimental conditions and properties of the medium for ξ_0 = 1. It is seen that the values of the period and the damping decrement decrease in the course of vibrations for fluids with $m > 1$ and increase for fluids with $m < 1$. These qualitative features can be explained in the following manner. According to the viscosimeter equation (10), the values of λ and δ grow with A for m = 1. For a nonlinear viscous medium we can take $A_{\text{n–v}} = A\sqrt{bD^{m-1}}$ as the parameter A. In the course of vibrations, the value of D averaged over a half-period drops and the apparent viscosity bD^{m-1} decreases, i.e., the quantity $A_{\text{n–v}}$ drops for dilatant media ($m > 1$) and grows for pseudoplastic ones ($m < 1$). The vibration parameters (11) accordingly vary with time. When m = 1 the value of $A_{\text{n–v}}$ and hence the values of λ and δ remain constant in the course of vibrations. We recall that dependence (10), describing the regular regime of vibrations, does not allow for transient processes.

The horizontal lines in Figs. 7b and 8 correspond to the analytical solution for the Newtonian medium (10) (b = 1 is the upper line and b = 0.1 is the lower line in Fig. 7b). When b = 1 the $\lambda = \lambda(N)$ curves tend to a single value of λ_1 for $N \to 1$ and different m. This enables us to determine A from dependence (10), as for m = 1, and to evaluate K under the assumption b = 1. When $b < 1$ the values of the vibration parameters $\lambda_1 = \lambda|_{N\to 1}$ and $\delta_1 = \delta|_{N\to 1}$ for nonlinear viscous media are higher than those for a Newtonian fluid, whereas when $b > 1$ they are lower

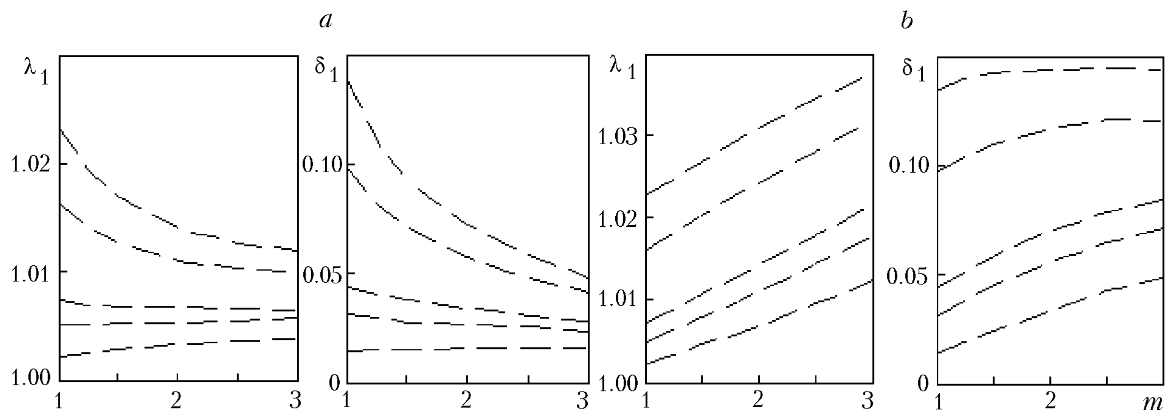

Fig. 9. Variation parameters λ_1 and δ_1 for different properties of the medium (A = 0.01; ξ_0 = 1 (a) and ξ_0 = 10 (b)): the downward curves correspond to b = 10, 5, 1, 0.5, and 0.1.

(when b = 10 we have the parameters δ = 0.137 and λ = 1.023 for the medium with m = 1). For the Newtonian fluid when $b \neq 1$, the vibration parameters have been determined from (10) with account for relations (8) for b and A.

Identification of Unknown Rheological Properties. The exponent of the power rheological law can be found by investigation of the asymptotic values of the parameters δ and λ for $N \to \infty$, when their variation with time is already slight and does not introduce a considerable error in view of the insufficient accuracy of measurement of these parameters for an individual vibration in practice. Another procedure of evaluation of unknown rheological properties is based on observation of the time behavior of the vibration parameters as a function of the vibration amplitude ξ_0. Below, we dwell in greater detail on another method of preliminary evaluation of b and m by the values of λ_1 and δ_1 for media with $m > 1$.

Curves demonstrating the change in λ_1 and δ_1 as a function of the medium's properties are given in Fig. 9 and are constructed with allowance for transient processes realized with initial conditions (7). The character of behavior of the vibration parameters as a function of m for different ξ_0 is determined by the type of flow, in particular, by the value of the apparent viscosity dependent on the modulus of the rate of shear. Performing experiments for a fixed value of A = 0.01, we can initially evaluate b and m by Fig. 9. These values are subsequently refined by comparison of the total dependences of the vibration parameters on time in the course of vibrations $\delta = \delta(N)$ and $\lambda = \lambda(N)$, i.e., by minimization of a quality function, which is the criterion of consistency of experimental and calculated data and is constructed, for example, by the least-squares method:

$$f(m, b) = \sum_{n=1}^{N} (\psi_{cn} - \psi_{en})^2 , \tag{12}$$

where $\psi = [\delta, \lambda]$. As the numerical experiments have known, the function (12) has a curvilinear ravine on the plane (m, b); therefore, here we must use ravine methods of search of a nonlocal character, for example, the configuration method. Also, it is noteworthy that we can generally take the vector $\psi = \xi$, i.e., can consider the consistency of experimental and calculated laws of vibrations. To improve the accuracy of measurement of the nonlinear properties we select the optimum experimental conditions by searching for the maximum of $\sum_n [(\partial\psi_n/\partial b)^2 + (\partial\psi_n/\partial m)^2]^{0.5}$ in the space of the unit's parameters and vibrations, and the maximum possible number of measurement points is taken with the equality of variances at different experimental points.

Conclusions. Thus, in the present work, for the regimes of forced and free damped vibrations:

(1) we have discussed the new possibilities of the vibrational method for observation of the nonlinear properties and identification of the rheological belonging of fluids to nonlinear viscous ones with a power rheological law;

(2) under conditions inherent in the vibrational method, we have revealed the effects associated with the behavior of such media and have studied their influence on the parameters observed in experiment; for the regime of forced vibrations, we have analyzed the influence of the nonlinear properties of fluids on the amplitude-phase characteristics under different experimental conditions and have investigated the appearance of odd harmonics of the driving force in the spectrum of kinematic and dynamic parameters of the system; for the regime of damped vibrations, we have shown that the properties of isosynchronism of vibrations are disturbed in motion of a plate in nonlinear viscous media;

(3) the distinctive features found for the plate's motion form the basis of the proposed methods of measurement of the nonlinear properties of fluid media by the characteristics of its law of vibrations.

The evaluations obtained for the change in the parameters of vibrational processes as a function of the rheological properties of media show that these changes can substantially exceed the accuracy of measurements, which enables us to hope that the effects associated with the non-Newtonian behavior of media are observable and their properties are identifiable in full-scale experiments, too.

NOTATION

A, dimensionless number characterizing experimental conditions and the system's properties; $A_{\text{n-v}}$, analog of A for nonlinear viscous media; a, dimensionless amplitude of vibrations of the plate; a_{as}, value of a_{res} for higher values of λ_{res}, a_{res}, amplitude of vibrations of the plate, corresponding to λ_{res}; b, dimensionless index of consistency; d, boundary-layer thickness, m; D, dimensionless second invariant of the strain-rate tensor; D_0, dimensionless rate of shear on the plate; $D_{\zeta\xi}$, dimensionless $\zeta\xi$th component of the strain-rate tensor; F, driving-force amplitude, N; f, quality function; I, intensity of spectral harmonics; i, imaginary unit ($i = \sqrt{-1}$); K, constant of the power rheological law, kg·sec^{m-2}/m; k, rigidity of the elastic element, kg/sec^2; M, mass of the suspended system, kg; m, exponent of the power rheological law; N, No. of vibration; n, No. of experimental point; S, surface area of the plate, m^2; T, dimensionless time; t, time, sec; U, dimensionless velocity; V, component of the plate velocity, directed along the X axis, m/sec; X, axis of the Cartesian coordinate system that is in parallel to the plate's plane; x, linear displacement of the plate from the equilibrium position, m; y, factor of proportionality between the scales along the X and Z axes; Z, axis of the Cartesian coordinate system that is orthogonal to the plate's plane; z, coordinate along the Z axis ($z = 0$ on the plate), m; β, dimensionless viscosity [1]; Δ, damping coefficient; Δt, difference between two neighboring instants of time, when $\xi = \xi(t)$ vanishes, sec; δ, logarithmic decrement of damping of vibrations; δ_1, value of δ for $N \to 1$; ζ, dimensionless coordinate; θ, reciprocal of λ; λ, dimensionless frequency; λ_1, value of λ for $N \to 1$; λ_{res}, dimensionless resonant frequency; μ, additional mass of the fluid, kg; ν, kinematic viscosity of the medium, m^2/sec; ξ, dimensionless linear displacement of the plate from the equilibrium position; ξ_0, dimensionless initial displacement of the plate; ξ_1 and ξ_2, neighboring extremum values of ξ ($|\xi_1| > |\xi_2|$); ρ, density of the medium, kg/m^3; $\sigma_{\zeta\xi}$, dimensionless $\zeta\xi$th component of the stress tensor; τ, period of vibrations of the plate immersed in the medium under study, sec; τ_0, period of natural vibrations, sec; φ, phase shift of vibrations of the plate from the exciting-force phase, deg; Φ_{fr}, dimensionless friction force acting on the plate; ψ_{c} and ψ_{e}, calculated and experimental values of the quantities measured in the experiment; ω, vibration frequency of the plate immersed in the medium under study, in particular, driving-force frequency in the method of forced vibrations, 1/sec; ω_0, frequency of natural vibrations of the plate, 1/sec. Subscripts: 0, parameters of natural vibrations and initial boundary conditions: $\xi_0 = \xi$ at $T = 0$ and $d_0 = D_{\zeta\xi}$ at $\zeta = 0$; 1, values of the parameters of vibrations for $N \to 1$; 1 and 2, enumeration order for ξ; n, index of summation; $\zeta\xi$, $\zeta\xi$th component; as, asymptotic; n-v, nonlinear viscous; c, calculated; res, resonant; fr, friction; e, experimental.

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